Mathisson-Papapetrou Equations as Conditions for the Compatibility of General Relativity and Continuum Physics

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Abstract

In continuum physics is presupposed that general-relativistic balance equations are valid which are created from the Lorentz-covariant ones by application of the equivalence principle. Consequently, the question arises, how to make these general-covariant balances compatible with Einstein's field equations. The compatibility conditions are derived by performing a modified Belinfante-Rosenfeld symmetrization for the non-symmetric and not divergence-free general-relativistic energy-momentum tensor. The procedure results in the Mathisson-Papapetrou equations.

In General Relativity Theory (GRT), as a consequence of Einstein's equations

$$R^{ab} - \frac{1}{2}g^{ab}R \;=\; \kappa\Theta^{ab} \quad\Longrightarrow\quad \Theta^{ab} \;=\; \Theta^{ba}, \quad \Theta^{ab}{}_{;b} \;=\; 0, \tag{1}$$

the energy-momentum tensor Θ^{ab} has to be symmetric and divergence-free[1]. Starting out with an action principle for deriving (1), Θ is the metric energy-momentum tensor[2] defined by

$$\Theta^{ab} \;:=\; \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_{mat}}{\delta g_{ab}}, \qquad \text{with} \quad \mathcal{L}_{mat} \;=\; \mathcal{L}_{mat}(g_{ab,c}, \Phi^A, \Phi^A_{,a}), \tag{2}$$

the Lagrange density of that matter-field Φ^A which is the source of the gravitational field described by the metric g_{ab}, determined by the solution of (1).

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[1]We use the definitions of the curvature and Ricci tensors given in [1]; the comma denotes partial and the semicolon covariant derivatives.

[2]This tensor can also be derived by exploiting the properties of the diffeomorphism group [2].

To be in accordance with the Einstein principle of equivalence, the general-covariant tensor Θ^{ab} should recover the corresponding Lorentz-covariant energy-momentum tensor of the matter in a local-geodesic coordinate system. Only symmetric and divergence-free Lorentz-covariant energy-momentum tensors can be transfered to a general-covariant tensor which can be used in the field equations (1) as a source of matter.

One starts with a Lorentz-covariant canonical energy-momentum tensor known from Special Relativity Theory (SRT) stemming from $\mathcal{L}_{mat}(\eta_{ab}, \Phi^A, \Phi^A_{\ ,a})$

$$\mathcal{T}^a_{\ b} \ := \ \frac{\partial \mathcal{L}_{mat}}{\partial \Phi^A_{\ ,a}} \Phi^A_{\ ,b} - \delta^a_b \mathcal{L}_{mat} \tag{3}$$

which in general is non-symmetric[3] and divergence-free

$$\mathcal{T}^{ab} \ \neq \ \mathcal{T}^{ba}, \qquad \mathcal{T}^{ab}_{\ \ ,a} \ = \ 0. \tag{4}$$

This tensor has to be symmetrized. Without symmetrization procedure, $\mathcal{T}^{ab}$ could not be used as matter-source term on the right-hand side of (1).

If matter-field equations can be derived by an action principle, balance equations for the spin are implied by the Bianchi identities[4]. In this sense, energy-momentum and spin balance are dependent on each other: they stem from the same origin. Here, our point of view is more pragmatic and less axiomatic: it concerns the fact that the phenomenological realm of application of GRT is wider than that one sketched above[5]. Often, one has neither a matter Lagrangian (or another specified matter model) nor does the energy-momentum tensor satisfy the conditions $(1)_{2,3}$. Then, in a special-relativistic version, one has phenomenological balance equations of the following type for energy-momentum and spin

$$T^{bc}_{\ \ ,b} \ = \ k^c, \qquad \text{with} \quad T^{bc} \ \neq \ T^{cb},^{6} \tag{5}$$

and

$$S^{cba}_{\ \ \ ,c} \ = \ m^{ba} \qquad \text{with} \quad S^{cba} \ = \ -S^{cab} \quad \text{and} \quad m^{ba} \ = \ -m^{ab}. \tag{6}$$

For non-isolated systems, $k^c \neq 0$ denotes an external force density, $m^{[ab]}$ is an external momentum density, and S^{cba} the current of spin density[7]. In particular, one finds such a situation in special-relativistic continuum thermodynamics, where the balances (5) must be supplemented by the balance

[3]This tensor is constructed by applying the Noether theorem. In [3], it is argued that the Noether procedure can also be performed resulting in a symmetric tensor. In this case, a symmetrization procedure is not needed.

[4]see, e.g., references [4, 5]

[5]If there is no Lagrange density –perhaps unknown or not existing– in continuum physics energy-momentum balance and spin balance have to be formulated separately and independently of each other.

[6]If there is a Lagrange density for the considered matter, then one has $T^{ab} = \mathcal{T}^{ab}$.

[7]often shortly denoted as spin tensor

equations of particle number and entropy density[8]. To consider these balance equations within GRT one has to rewrite them in a general-covariant form:

$$T^{bc}{}_{;b} \;=\; k^c, \qquad S^{cba}{}_{;c} \;=\; m^{ba}, \tag{7}$$

Now, the question arises: How can these balance equations be incorporated beside the gravitational equations (1) into the general-covariant framework of GRT, without getting in contradiction[9]? In the present brief note, we ask for these conditions by which a general-relativistic construction of a symmetric energy-momentum tensor from the non-symmetric T^{bc} is possible, so that in a local-geodesic coordinate system the relation between these tensors reduces to their special-relativistic relation given by the Belinfante-Rosenfeld symmetrization [8].

We start out for remembering with a sketch of the usual Belinfante-Rosenfeld symmetrization applied to the canonical energy-momentum tensor (4) [8]. Defining a hyper-potential Σ^{abc}

$$\Sigma^{abc} \;:=\; S^{abc} + S^{bca} + S^{cba}, \qquad S^{abc} \;=\; -S^{acb}\ \text{[10]} \qquad \Sigma^{abc} \;=\; -\Sigma^{bac}. \tag{8}$$

Belinfante and Rosenfeld define the tensor

$$B^{bc} \;:=\; \not{T}^{bc} - \frac{1}{2}\Sigma^{abc}{}_{,a}. \tag{9}$$

Since in SRT by use of the Lagrange density $(2)_2$, S^{abc} is fixed and

$$\Sigma^{a[bc]}{}_{,a} \;\equiv\; S^{abc}{}_{,a} \;=\; 2\,\not{T}^{[bc]} \quad\Longrightarrow\quad B^{[bc]} \;=\; 0 \tag{10}$$

holds true, B^{bc} is symmetric. Furthermore, because of the vanishing divergence of the canonical energy-momentum tensor and because of $(8)_3$ and commuting partial derivatives, one obtains

$$\Sigma^{abc}{}_{,a,b} \;=\; 0 \quad\Longrightarrow\quad B^{bc}{}_{,b} \;=\; 0 \tag{11}$$

that the divergence of B^{bc} vanishes. Consequently, one obtains the desired special-relativistic relations

$$B^{[bc]} \;=\; 0, \qquad B^{bc}{}_{,b} \;=\; 0. \tag{12}$$

We now investigate, if such a symmetrization procedure can also operate in GRT. In contrast to the usual procedure, we do not take the general-covariantly rewritten symmetrized tensor B^{bc} in (9), satisfying (12), as source of Einstein's equations, but we set out with the general-covariantly rewritten

[8] For this continuum theory of irreversible processes, see the contributions in [4] and [6].

[9] as e.g. assumed in [7]

[10] Here S^{abc} is given by the Lagrange density $\mathcal{L}_{mat}(\eta_{ab}, \Phi^A, \Phi^A_{,a})$.

full tensor $(5)_1$, that means, with T^{bc} instead of $\not{T}^{bc}$. Analogously to (8), we make the following ansatz

$$
\begin{aligned}
{}^\dagger\Sigma^{abc} &:= {}^\dagger S^{abc} + {}^\dagger S^{bca} + {}^\dagger S^{cba}, && (13)\\
{}^\dagger S^{abc} &= -{}^\dagger S^{acb}, \qquad {}^\dagger\Sigma^{abc} = -{}^\dagger\Sigma^{bac}. && (14)
\end{aligned}
$$

Except for the anti-symmetry in the two last indices, ${}^\dagger S^{abc}$ is not specified so far. Then, motivated by the Belinfante-Rosenfeld procedure, we define

$$
{}^\dagger B^{bc} := T^{bc} - \frac{1}{2}{}^\dagger\Sigma^{abc}{}_{;a}, \tag{15}
$$

Because this tensor does not automatically satisfy (12), we now have to demand that ${}^\dagger B^{bc}$ has to be symmetric and divergence-free:

$$
\begin{aligned}
{}^\dagger B^{[bc]} &\stackrel{\bullet}{=} 0 \quad\Longrightarrow\quad {}^\dagger\Sigma^{a[bc]}{}_{;a} \equiv {}^\dagger S^{abc}{}_{;a} = 2T^{[bc]}, && (16)\\
{}^\dagger B^{bc}{}_{;b} &\stackrel{\bullet}{=} 0 \quad\Longrightarrow\quad T^{bc}{}_{;b} - \frac{1}{2}{}^\dagger\Sigma^{abc}{}_{;a;b} = 0. && (17)
\end{aligned}
$$

For calculating ${}^\dagger\Sigma^{abc}{}_{;a;b}$, we start out with the relation for the second covariant derivative taking the symmetry properties of Σ^{abc} and those of the curvature tensor $R^a{}_{bc}$ into account

$$
\begin{aligned}
2{}^\dagger\Sigma^{abc}{}_{;a;b} &= {}^\dagger\Sigma^{abc}{}_{;a;b} - {}^\dagger\Sigma^{abc}{}_{;b;a} =\\
&= R^a_{mab}{}^\dagger\Sigma^{mbc} + R^b_{mab}{}^\dagger\Sigma^{amb} + R^c_{mab}{}^\dagger\Sigma^{abm} =\\
&= R_{mb}{}^\dagger\Sigma^{mbc} - R_{ma}{}^\dagger\Sigma^{amb} + R^c_{mab}{}^\dagger\Sigma^{abm} = R^c_{mab}{}^\dagger\Sigma^{abm}. && (18)
\end{aligned}
$$

This results in

$$
\begin{aligned}
2{}^\dagger\Sigma^{abc}{}_{;a;b} &= -\Big(R^c_{abm} + R^c_{bma}\Big){}^\dagger\Sigma^{abm} = -R^c_{abm}{}^\dagger\Sigma^{a[bm]} - R^c_{bam}{}^\dagger\Sigma^{bam} =\\
&= -R^c_{abm}{}^\dagger\Sigma^{a[bm]} - R^c_{bam}{}^\dagger\Sigma^{b[am]} = -2R^c_{abm}{}^\dagger\Sigma^{a[bm]}. && (19)
\end{aligned}
$$

Taking $(17)_2$ into account, we obtain

$$
T^{bc}{}_{;b} = -\frac{1}{2}R^c_{abm}S^{abm}. \tag{20}
$$

This demonstrates that the required compatibility of the relations (7) with Einstein's field equations is guaranteed when the Mathisson-Papapetrou equations $(16)_3$ and (20)are satisfied[11]. In other words, for the sake of compatibility the external force density and the external momentum density must be specified as follows

$$
k^c := -\frac{1}{2}R^c_{abm}S^{abm}, \qquad m^{[bc]} := 2T^{[bc]}. \tag{21}
$$

[11]These equations were first derived for pole-dipole particles by Mathisson [9] and Papapetrou [10]. Later they were also proved to be true for the free motion of continua with an intrinsic classical spin S^{abc}, where $S^{abc} = u^a S^{bc}$. First time, this was done by Weyssenhoff and Raabe [11] for the special model of an ideal fluid with spin and, afterwards, for models specified by the choice of the Lagrangian, as in [5, 12].

In continuum physics of SRT is common use, that energy-momentum and spin can be balanced by (5) and (6). If one presupposes that in GRT the general-covariantly written balances (7) of energy-momentum and spin hold true analogously to the SRT, then the external forces and the external moments are specified by the gravitational field as given in (21): energy-momentum and spin balances transfer into the Mathisson-Papapetrou equations. The external sources do not vanish in GRT due to the gravitational field, even if the energy-momentum tensor is symmetric. Consequently, the Mathisson-Papapetrou equations are the basic equations of general-relativistic continuum physics, if the validity of (7) is presupposed.

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